DOI: 10.2478/auom-2020-0029
An. Şt. Univ. Ovidius Constanţa **Vol. 28(2),2020, 229–237**

On some mean value theorem via covering argument

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Abstract

We show how the full covering argument can be used to prove some type of Cauchy mean value theorem.

1 Introduction

The full covering argument based on the Cousin's lemma (see, e.g., [1], [2], [9], [11]) allows as to obtain very elegant and clear proofs of many classical results from mathematical analysis (see [1], [2], [6], [10], [11, Sections 4.5.3, 4.5.4]). Recall that

Definition 1. A family $\mathcal{C} \subset \{(x, [s,t]) : \ x \in [s,t] \subset [a,b]\}$ is a *full cover* of an interval $[a,b]$, if there exists a function $\delta \colon [a,b] \to (0,\infty)$ such that $(x,[c,d]) \in \mathcal{C}$ for $x \in [a,b]$ and $a \leq c \leq x \leq d \leq b$ with $0 < d - c < \delta(x)$.

Definition 2. By a *partition* of an interval $[a,b]$ we mean a finite family $\{(x_k, [z_{k-1}, z_k]) : \ k = 1, \ldots, n\}$ such that $x_k \in [z_{k-1}, z_k]$ for every $k \in \{1, \ldots, n\}$ and $a = z_0 < z_1 < \ldots < z_n = b$.

The crucial in this theory, mentiond above result reads as follows:

Key Words: full cover, right adequate cover, partition, mean value theorem.
2010 Mathematics Subject Classification: Primary 26A06, 26A24.
Received: 03.09.2019
Revised: 25.09.2019
Accepted: 09.10.2019

Lemma 1 (Cousin)**.** *Every full cover of an interval* $[a,b]$ *contains a partition of* $[a,b]$.

To show the standard argumentation with the use of full covers, we prove for example the following result (cf. [1, Theorem 1]):

Proposition. *If the function* $f\colon [a,b] \to \mathbb{R}$ *is lower* (*resp., upper*) *semicontinuous on* $[a,b]$*, then* f *is bounded from below* (*resp., from above*) *on* $[a,b]$.

Proof. Assume that f is lower semicontinuous and put

$$\mathcal{C} := \big\{(x,[c,d]) : \ x \in [c,d] \subset [a,b] \text{ and } f \text{ is bounded from below on } [c,d]\big\}.$$

We show that $\mathcal{C}$ is a full cover of $[a,b]$. Fix $x \in [a,b]$ and $\varepsilon > 0$. Since f is lower semicontinuous at x, there exists $\delta(x) > 0$ such that

$$f(y) > f(x) - \varepsilon \quad \text{for } y \in (x-\delta(x), x+\delta(x)).$$

So, if we take an interval $[c,d] \subset [a,b]$ such that $x \in [c,d]$ and $d-c < \delta(x)$, then $[c,d] \subset (x-\delta(x), x+\delta(x))$ and, consequently, f is boundend from below on $[c,d]$ (by a constant $f(x)-\varepsilon$).

Due to Cousin's lemma we can find a partition $\{(z_1,I_1),\dots,(z_n,I_n)\} \subset \mathcal{C}$ of $[a,b]$. Since f is bounded from below on every set I_k, for every $k \in \{1,\dots,n\}$ there exists $m_k \in \mathbb{R}$ such that

$$f(x) > m_k \quad \text{for } x \in I_k.$$

Thus, using the fact that $[a,b] = I_1 \cup \ldots \cup I_n$, we get

$$f(x) > m \quad \text{for } x \in [a,b],$$

where $m := \min_{1\le k\le n} m_k$. □

Using full covers one can, in particular, define the Henstock integral generalizing the Riemann integral and show its many properties (see, e.g., [4], [8]). In 2006, J.W. Hagood and B.S. Thomson ([5]) generalized the Henstock integral. For this purpose they introduced the notion of a *right adequate cover* generalizing a full covering relation and proved a counterpart of the Cousin's lemma connected with this new relation ([5, Lemma 7]).

In this note we give a new proof of some version of the Cauchy mean value theorem using the concept of a right adequate cover and based on the ideas of papers [6] and [5] (see Theorem below), then we comment some possible versions of our theorem and its connections with other results obtained with the same technique, and finally we formulate a few classical corollaries which easy follow from our result.

2 Main result

If the functions $f, g\colon [a,b] \to \mathbb{R}$ are continuous on $[a,b]$, differentiable in (a,b) and $g'(x) \neq 0$ for $x \in (a,b)$, then according to the basic Cauchy mean value theorem there exists $c \in (a,b)$ such that $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$. Assuming now additionaly that $f'(x) \leq g'(x)$ (resp., $|f'(x)| \leq g'(x)$) for $x \in (a,b)$, as a direct consequence of the Cauchy result we get the inequality $f(b) - f(a) \leq g(b) - g(a)$ (resp., $|f(b) - f(a)| \leq g(b) - g(a)$). The mean value theorem with this mean value inequality was generalized by many mathematicians at least in two directions: by replacing standard derivatives by right-hand-side derivatives as well as by considering functions with values in more general spaces (e.g., in Banach spaces) – see, e.g., [3, Theorem 8.5.1, Problem 8.5.2], [7, Theorem B]. In the present short note we give the new proof to one of such results by using covering relation. This technique was used in the proofs of the following mean value theorems: [2, Theorems 7,8], [6, Theorems 1,2] and [5, Theorem 9]. Similar to this last theorem we consider right-hand derivatives, and similar to [6, Theorem 1] we take functions with complex values.

Recall that by $g'_+(x)$ we denote the right-hand-side derivative:

$$g'_+(x) = \lim_{y \to x+} \frac{g(y) - g(x)}{y - x}.$$

We consider also so called *right-hand derivative values* (cf. [7]). Namely, if $f\colon [a,b] \to \mathbb{C}$ and $x \in [a,b)$ then an element $y \in \mathbb{C}$ is a right-hand derivative value of f at x if there exists a decreasing sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n \to x$ and

$$y = \lim_{n \to \infty} \frac{f(t_n) - f(x)}{t_n - x}.$$

Definition 3. [5, Definition 6] We call a family $\mathcal{C} \subset \{(x, [s,t]) :\ x \in [s,t] \subset [a,b]\}$ a *right adequate cover* of an interval $[a,b]$, if there exist two functions $r\colon [a,b) \to (a,b)$, $l\colon (a,b] \to (a,b)$ such that $l(x) < x < r(x)$ for $x \in (a,b)$,

- $(a, [a, r(a)]) \in \mathcal{C}$,
- $(b, [s, b]) \in \mathcal{C}$ for $s \in (l(b), b)$,
- $(x, [s, r(x)]) \in \mathcal{C}$ for $x \in (a,b)$, $s \in (l(x), x]$.

Lemma 2. ([5, Lemma 7]) *Every right adequate cover of an interval* $[a,b]$ *contains a partition of* $[a,b]$.

Applying the above lemma we obtain the following generalization of [2, Theorem 7], [6, Theorems 1,2 (A),(B)]:

Theorem. *Let $f\colon [a,b] \to \mathbb{C}$ (resp., $f\colon [a,b] \to \mathbb{R}$) be a function continuous from the left on $(a,b]$, $g\colon [a,b] \to \mathbb{R}$ a nondecreasing function and Q at most countable subset of $[a,b]$. Assume that for every $x \in [a,b) \setminus Q$ there exists a right-hand derivative value $RDf(x)$ of f such that*

$$|RDf(x)| \le g'_+(x) < \infty \tag{1}$$

$$(resp., \quad RDf(x) \le g'_+(x) < \infty),$$

g is continuous from the left on $(a,b] \setminus Q$ and for every $x \in Q$ there exists a sequence $(y_n)_{n\in\mathbb{N}}$ such that $y_n \in (x,b]$, $n \in \mathbb{N}$, $\lim_{n\to\infty} y_n = x$ and $\lim_{n\to\infty} f(y_n) = f(x)$. Then

$$|f(b) - f(a)| \le g(b) - g(a) \quad (resp., \ f(b) - f(a) \le g(b) - g(a)).$$

Proof. Let $Q = \{y_n :\ n \in N\}$, where $N \subset \mathbb{N}$. Fix $\varepsilon > 0$ and set

$$\begin{aligned}
\mathcal{D} &:= \big\{(x,[u,v]) :\ x \in [u,v] \setminus Q \subset [a,b), \\
&\qquad\qquad\qquad\qquad |f(v) - f(u)| \le g(v) - g(u) + \varepsilon(v-u)\big\}, \\
\mathcal{E} &:= \big\{(b,[u,b]) :\ a \le u < b,\ |f(b) - f(u)| \le g(b) - g(u) + \varepsilon(b-u+1)\big\}, \\
\mathcal{F}_n &:= \big\{(y_n,[u,v]) :\ y_n \in [u,v] \subset [a,b),\ |f(v) - f(u)| \le \varepsilon 2^{-(n+1)}\big\},\ n \in N.
\end{aligned}$$

We show that the family

$$\mathcal{C} := \mathcal{D} \cup \mathcal{E} \cup \bigcup_{n\in N} \mathcal{F}_n$$

is a right adequate cover of $[a,b]$. Fix $x \in [a,b]$.

If $x = a \notin Q$ then we can take a number $\delta > 0$ such that

$$\left| \frac{g(v) - g(a)}{v-a} - g'_+(a) \right| \le \frac{\varepsilon}{2} \quad \text{for } v \in (a, a+\delta) \cap (a,b),$$

and then choose a number $r(a) \in (a,b) \cap (a,a+\delta)$ for which

$$\left| \frac{f(r(a)) - f(a)}{r(a) - a} - RDf(a) \right| \le \frac{\varepsilon}{2}.$$

Hence,

$$|f(r(a)) - f(a)| \le |RDf(a)|(r(a) - a) + \frac{\varepsilon}{2}(r(a) - a),$$

$$g'_+(a)(r(a) - a) \le g(r(a)) - g(a) + \frac{\varepsilon}{2}(r(a) - a),$$

which in view of (1) implies the inequality

$$|f(r(a)) - f(a)| \le g(r(a)) - g(a) + \varepsilon(r(a) - a)$$

and consequently, $(a, [a, r(a)]) \in \mathcal{D} \subset \mathcal{C}$.

If $x \in (a, b) \setminus Q$ then we choose at first $\delta > 0$ such that

$$\left| \frac{g(v) - g(x)}{v - x} - g'_+(x) \right| < \frac{\varepsilon}{2} \quad \text{for } v \in (x, x + \delta) \cap (a, b),$$

and then a number $r(x) \in (x, x + \delta) \cap (a, b)$ such that

$$\left| \frac{f(r(x)) - f(x)}{r(x) - x} - RDf(x) \right| < \frac{\varepsilon}{2}.$$

In particular, we have

$$\frac{|f(r(x)) - f(x)|}{r(x) - x} < |RDf(x)| + \frac{\varepsilon}{2}, \quad g'_+(x) < \frac{g(r(x)) - g(x)}{r(x) - x} + \frac{\varepsilon}{2}.$$

Since f and g are continuous from the left at x, we can find $l(x) \in (a, x)$ such that

$$\frac{|f(r(x)) - f(u)|}{r(x) - u} < |RDf(x)| + \frac{\varepsilon}{2} \quad \text{and} \quad g'_+(x) < \frac{g(r(x)) - g(u)}{r(x) - u} + \frac{\varepsilon}{2}$$

for $u \in (l(x), x]$. Applying now (1) we get

$$\begin{aligned} |f(r(x)) - f(u)| &< |RDf(x)|(r(x) - u) + \frac{\varepsilon}{2}(r(x) - u) \\ &\le g'_+(x)(r(x) - u) + \frac{\varepsilon}{2}(r(x) - u) \\ &< g(r(x)) - g(u) + \varepsilon(r(x) - u), \end{aligned}$$

thus, $(x, [u, r(x)]) \in \mathcal{D} \subset \mathcal{C}$ for $u \in (l(x), x]$.

Assume now that $x = b$. Using continuity from the left of both functions at b we can choose $l(b) \in (a, b)$ such that

$$|f(b) - f(u)| \le g(b) - g(u) + \varepsilon(b - u + 1) \quad \text{for } u \in (l(b), b),$$

i.e., $(b, [u, b]) \in \mathcal{E} \subset \mathcal{C}$ for $u \in (l(b), b)$.

If $x = a = y_n$ for some $n \in N$ then by our assumption there is a number $r(a) \in (a, b)$ for which

$$|f(r(a)) - f(a)| \le \varepsilon 2^{-(n+1)},$$

i.e., $(a, [a, r(a)]) \in \mathcal{F}_n \subset \mathcal{C}$.

Finally, if $x = y_n \in (a, b)$ for some $n \in N$ then there exist numbers $a < l(x) < x < r(x) < b$ such that

$$|f(r(x)) - f(u)| \leq \varepsilon 2^{-(n+1)} \quad \text{for } u \in (l(x), x],$$

which means that $(x, [u, r(x)]) \in \mathcal{F}_n \subset \mathcal{C}$ for $u \in (l(x), x]$.

By Lemma 2 there exists a partition $\{(x_k, [z_{k-1}, z_k] : \ k = 1, \dots l\} \subset \mathcal{C}$ of $[a, b]$. Note that $x_l = b$ and every family $\mathcal{F}_n$ contains at most two elements of our partition (exactly two, if $y_n = x_k$ for some $k \in \{1, \dots, l\}$ and it is a label of adjacent intervals). Consequently, using also monotonicity of g, we have

$$\begin{aligned}
|f(b) - f(a)| \leq & \sum_{i=1}^{l} |f(z_i) - f(z_{i-1})| = \sum_{x_i \in [a,b) \setminus Q} |f(z_i) - f(z_{i-1})| \\
& + \sum_{n \in N} \sum_{x_i = y_n} |f(z_i) - f(z_{i-1})| + |f(b) - f(z_{l-1})| \\
\leq & \sum_{x_i \in [a,b) \setminus Q} \big(g(z_i) - g(z_{i-1}) + \varepsilon(z_i - z_{i-1})\big) + \sum_{n \in N} 2 \frac{\varepsilon}{2^{n+1}} \\
& + g(b) - g(z_{l-1}) + \varepsilon(b - z_{l-1} + 1) \\
\leq & \ g(b) - g(a) + \varepsilon(b - a + 2).
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we obtain the desired result.

In the case of real-valued function f the proof goes by the same way with only slight modifications (cf. also [5, Theorem 9] and its proof). □

3 Comments

Thoroughly analysing the above proof one can formulate the following observations concerning our theorem:

1. The assumption of monotonicity of g is used only in the last estimation and if $Q = \emptyset$ it can be omited.

2. In case of the function f with real values we can replace its left-hand-side continuity by the weaker assumption: $\liminf_{y \to x-} f(y) \geq f(x)$ for $x \in (a, b]$ (cf. also [5, Section 5.a]).

3. In case of the function f taking its values in $\mathbb{C}$ we can replace this set by a normed vector space and, simultaneously, the absolute value by a

norm (cf. also the last paragraph in [2] and the paragraph under [6, Theorem 1]). Even more, we can consider the function f with values in a topological linear space assuming at the same time the inequality $p(DRf(x)) \leq g'_+(x)$ instead of (1), where p is a real continuous subadditive and positively homogeneous functional on this space – then we get the inequality $p(f(b) - f(a)) \leq g(b) - g(a)$ (cf. [7, Theorem B]).

4. The assumption (1) (resp., its counterpart in real case) can be replaced by the following: $|RDf(x)| \leq RDg(x)$ (resp., $RDf(x) \leq RDg(x)$), where right-hand derivative velues $RDf(x)$ and $RDg(x)$ are associated with the same decreasing sequence $(t_n)_{n\in\mathbb{N}}$ tending to x (cf. [7, Theorem B]).

The notion of the right adequate cover introduced in [5] allowed us to generalize [2, Theorem 7], [6, Theorems 1,2 (A),(B)] by considering one-sided derivatives only, and simultaneously to keep the standard, characteristic of covering relations method, argumentation. Our theorem and [5, Theorem 9] are independent, however this last theorem together with [5, Section 5.b] and [8, Theorem 4] also generalizes [6, Theorem 2 (A),(B)]. Moreover, our result together with the third comment generalizes [3, Theorem 8.5.1, Problem 8.5.2], and in case of continuous functions f and g it is a consequence of [7, Theorem B].

4 Corollaries

Finally, we formulate three classical results from real and complex analysis being immediate consequences of our main theorem. Recall that by $D_+f(x)$ and $D^+f(x)$ we denote right-hand Dini derivatives:

$$D_+f(x) = \liminf_{y\to x+} \frac{f(y)-f(x)}{y-x}, \quad D^+f(x) = \limsup_{y\to x+} \frac{f(y)-f(x)}{y-x}.$$

Applying Theorem with $Q = \emptyset$ and taking into account the first comment we get the following result:

Corollary 1. *If $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ are continuous from the left on $(a,b]$ and satisfy the inequality $-\infty < D_+f(x) \leq g'_+(x) < \infty$ (resp., $\infty > D^+f(x) \geq g'_+(x) > -\infty$) for $x \in [a,b)$, then $f(b) - f(a) \leq g(b) - g(a)$ (resp. $f(b) - f(a) \geq g(b) - g(a)$).*

Proof. In case of the inequality $-\infty < D_+f \leq g'_+ < \infty$ it is enough to use Theorem together with our first comment. If $\infty > D^+f \geq g'_+ > -\infty$ then it is enough to apply the first part of this corollary to $-f$, $-g$ and to observe that $D^+(-f) = -D_+ f$. $\square$

Corollary 2. (cf. [11, Theorem 7.30, Exercises 7.8.6–7.8.8]) *If $f\colon [a,b] \to \mathbb{R}$ is continuous from the left on $(a,b]$ and $-\infty < D_+ f(x) \le 0$ (resp., $\infty > D^+ f(x) \ge 0$) for $x \in [a,b)$, then f is nonincreasing (resp., nondecreasing).*

Proof. Assume that $D_+ f \le 0$ on $[a,b)$ and fix $c, d \in [a,b]$, $c < d$. By Corollary 1 used to the functions $f|_{[c,d]}$, $g \equiv 0$, we have $f(d) - f(c) \le 0$.

If $D^+ f \ge 0$ on $[a,b)$ then it is enough to apply the second part of our first corollary. $\square$

Corollary 3. *If $f\colon [a,b] \to \mathbb{C}$ is continuous from the left on $(a,b]$ and for every $x \in [a,b)$ zero is a right-hand derivative value of f at x, then f is constant.*

Proof. Fix $c \in (a,b]$. By Theorem used to the functions $f|_{[a,c]}$, $g \equiv 0$, we have $|f(c) - f(a)| \le 0$, i.e., $f(c) = f(a)$. $\square$

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Acknowledgment. This research was supported by the University of Silesia Mathematics Department (Iterative Functional Equations and Real Analysis program).

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